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LETTER TO THE EDITOR

Painlevé property and constants of the motion of the complex Lorenz model

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Abstract. The results of the Painlevé analysis and of a search for constants of the motion of the complex Lorenz model are presented.

In a series of papers [1–3] Fowler, Gibbon and McGuinness have undertaken a study of the complex Lorenz equations

$$\dot{x} = -\sigma x + \sigma y \tag{1}$$

$$\dot{y} = (r - z)x - ay \tag{2}$$

$$\dot{z} = -bz + \tfrac{1}{2}(x^* y + x y^*) \tag{3}$$

where x and y are complex and z is real, and the complex parameters r and a are defined by

$$r = r_1 + \mathrm{i} r_2 \tag{4}$$

$$a = 1 - \mathrm{i} e \tag{5}$$

and σ and b are real. These equations can be derived from the amplitude equations arising in a stability analysis of certain non-linear optical and hydrodynamical systems with weak dispersion and dissipation [2]. Recently, they were also shown to be relevant for the study of collisional inhomogenous plasmas [4]. In agreement with the physical interpretation we assume that r_1, σ and b are non-negative. The well known real Lorenz model is obtained from (1)–(3) by setting $r_2 = e = 0$ and restricting x and y to be real.

The bifurcation behaviour of (1)–(3) is remarkably different from the one of the real Lorenz model [1]. In particular, it was shown that for $r_1 > r_{1c}$ with

$$r_{1c} = 1 + \frac{(e + r_2)(e - \sigma r_2)}{(\sigma + 1)^2} \tag{6}$$

(1)–(3) has an exact periodic solution of the form

$$x = A \exp(\mathrm{i}\omega t)$$

$$y = A(1 + \mathrm{i}\omega/\sigma) \exp(\mathrm{i}\omega t)$$

$$z = |A|^2/b \tag{7}$$

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where

$$|A|^2 = b(r_1 - r_{1c}) \tag{8}$$

$$\omega = \frac{\sigma(e+r_2)}{\sigma+1} \tag{9}$$

which is a stable limit cycle for all $r_1 > r_{1c}$ provided $\sigma < b+1$.

On the other hand, Tabor and Weiss [5] have studied the location and the type of singularities that occur when the real Lorenz equations are made complex by the extension to complex time. They found that these complex-time singularties are critical in determining the behaviour of the real-time solution.

Here we consider the complex Lorenz equations for complex time. In practice, one can proceed by first rewriting (1)-(3) as a set of five real equations. Let $x = x_1 + \mathrm{i}x_4$, $y = x_2 + \mathrm{i}x_5$ and $z = x_3$; then (1)-(3) implies

$$\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\
\dot{x}_2 &= -x_2 - ex_5 - r_2 x_4 + (r_1 - x_3)x_1 \\
\dot{x}_3 &= -bx_3 + x_1 x_2 + x_4 x_5 \\
\dot{x}_4 &= -\sigma x_4 + \sigma x_5 \\
\dot{x}_5 &= -x_5 + ex_2 + r_2 x_1 + (r_1 - x_3)x_4.
\end{aligned} \tag{10}$$

Next (10) can be studied in the complex-time plane by the standard methods [5]. It is an open question what the complex-time singularities of (10) are, how they are related to the complex-time singularities of the real Lorenz equations and to what extent they are critical for the behaviour of the real-time solution of the complex Lorenz equations.

The first step in answering these questions concerns the determination of all cases for which (10) has only poles as movable singularities in the complex-time plane (Painlevé property) and the determination of cases for which (10) has (exponentially damped) polynomial conserved quantities. For the real Lorenz equations it was found [5] that there are three cases for which the system has the Painlevé property (we now assume $\sigma \neq 0$) and that in these cases the system has at least one polynomial conserved quantity. On the other hand, polynomial conserved quantities can exist in the absence of the Painlevé property. For the real Lorenz model all polynomial constants of the motion up to fourth order in the variables were obtained by Kus [6] and it was shown by Schwarz and Steeb [7] that his results are complete up to sixth order. In this letter we present the result of the Painlevé analysis and the search for constants of the motion of the complex Lorenz equations (1)-(3) or (10).

Straightforward application of the well known algorithm to perform the Painlevé analysis (see e.g. [8, 9]) reveals that the complex Lorenz equations have the Painlevé property in the following three cases:

(i) $\sigma = \frac{1}{2}$, $r_1 = \frac{1}{2}e^2$, $b = 1$, $r_2 = \frac{1}{2}e$, e arbitrary (11)

(ii) $\sigma = 1$, $r_1 = \frac{1}{4}e^2 + \frac{1}{9}$, $b = 2$, $r_2 = 0$, e arbitrary (12)

(iii) $\sigma = \frac{1}{3}$, r_1 arbitrary, $b = 0$, $r_2 = e$, e arbitrary. (13)

Using the REDUCE program DISSYS written by Schwarz [10], all constants of the motion of (10) of the form

$$F(x_1, x_2, x_3, x_4, x_5, t) = \exp(c_0 t) P(x_1, x_2, x_3, x_4, x_5) \tag{14}$$

where c_0 is a constant and P is a polynomial of at most fourth order in its arguments were determined. Six cases with P quadratic and three cases with P of fourth order were found. Expressed in terms of x, y and z all quadratic invariants are of the form

$$F(x, y, z, t) = \exp(c_0 t)[a_0 xx^* + a_1(yy^* + z^2) + a_2 z - \tfrac{1}{2}\mathrm{i}a_3(x^*y - xy^*) \tag{15}$$

where c_0, a_1, a_2 and a_3 are constants depending on σ, b, r_1, r_2 and e. The results are given in table 1 where (16), (17) and (18) respectively stand for

$$e = \sigma r_2/(1-\sigma) + r_1(1-\sigma)/r_2 \tag{16}$$

$$a_2 = (1-\sigma)e - 2\sigma r_2 \tag{17}$$

$$r_2 = (1-\sigma)e/2\sigma. \tag{18}$$

The three quartic invariants are all of the form

$$F(x, y, z, t) = \exp(c_0 t)[-(xx^*)^2 + a_4 xx^* z + a_5 xx^* + a_6 yy^* + \tfrac{1}{2}a_7(xy^* + x^*y) - \tfrac{1}{2}\mathrm{i}a_8(x^*y - xy^*) + a_9 z] \tag{19}$$

with c_0, a_4, a_5, a_6, a_7, a_8 and a_9 constants depending on σ, b, r_1, r_2 and e. The three cases respectively are

(i) $\sigma = 1 \qquad r_1 = \text{arbitrary} \qquad b = 4$

$r_2 = \text{arbitrary} \qquad e = \text{arbitrary}$

$c_0 = 4 \qquad a_4 = 4 \qquad a_5 = 4r_1 + 2r_2^2 + 2r_2 e$ (20)

$a_6 = 4 \qquad a_7 = -8 \qquad a_8 = -4(r_2 + e)$

$a_9 = 16 - 16r_1 - 4r_2^2 + 4e^2$

(ii) $\sigma = \text{arbitrary } (\neq \tfrac{1}{3})$

$r_1 = 2\sigma - 1 + (-r_2 + e)[\sigma r_2 + (2\sigma - 1)e]/(3\sigma - 1)^2$

$b = 6\sigma - 2 \qquad r_2 = \text{arbitrary} \qquad e = \text{arbitrary}$

$c_0 = 4\sigma \qquad a_4 = 4\sigma$ (21)

$a_5 = 4(2\sigma - 1)^2 + 4[\sigma r_2 + (2\sigma - 1)e]^2/(3\sigma - 1)^2$

$a_6 = 4\sigma^2 \qquad a_7 = -8\sigma(2\sigma - 1)$

$a_8 = -8\sigma[\sigma r_2 + (2\sigma - 1)e]/(3\sigma - 1) \qquad a_9 = 0$

(iii) $\sigma = \tfrac{1}{3} \qquad r_1 = \text{arbitrary}$

Table 1. Cases for which the complex Lorenz equations have a constant of the motion of the form (15).

	Parameter of the model					Parameters of (15)				
	σ	r_1	b	r_2	e	c_0	a_0	a_1	a_2	a_3
1	arbitrary	arbitrary	2σ	arbitrary	arbitrary	2σ	1	0	-2σ	0
2	arbitrary	0	1	0	arbitrary	2	0	1	0	0
3	1	arbitrary	1	0	arbitrary	2	$-r_1$	1	0	0
						2	$-\tfrac{1}{2}e$	0	0	1
4	arbitrary $\neq 1$	arbitrary	1	arbitrary $\neq 0$	(16)	2	$\tfrac{1}{2}r_2/(\sigma-1)$	$\tfrac{1}{2}(\sigma-1)/r_2$	0	1
5	arbitrary	arbitrary	$\sigma+1$	arbitrary	arbitrary	$\sigma+1$	r_2	0	(17)	$\sigma-1$
6	arbitrary	arbitrary	arbitrary	(18)	arbitrary	$\sigma+1$	$-e$	0	0	2σ

$$
\begin{array}{llll}
b=0 & r_2=\text{arbitrary} & e=r_2 & \\
c_0=\frac{4}{3} & a_4=\frac{4}{3} & a_5=-\frac{4}{3}r_1 & \\
a_6=\frac{4}{9} & a_7=\frac{8}{9} & a_8=0 & a_9=0.
\end{array}
\qquad (22)
$$

We conclude with a discussion. First, it should be remarked that the results for the real Lorenz model are embedded in a natural way in (11)-(22). Indeed, setting $e=0$ in (11)-(13) the parameters σ, r_1 and b have the value for which the real Lorenz equations have the Painlevé property. Furthermore, setting $r_2=e=0$ and restricting x and y to be real the first three quadratic invariants and the fourth-order invariants reduce to the known invariants of the real Lorenz model whereas the other invariants reduce to zero (or do not exist, case 4 in table 1). Correspondingly, when the complex Lorenz model has the Painlevé property it has as constants of the motion of the generalisations of the constants of the motion present in the associated Painlevé case of the real Lorenz model: Painlevé case (11) has the quadratic invariants 4 and 6 of table 1. Painlevé case (12) has the quadratic invariant 1 of table 1 and Painlevé case (13) has the fourth-order invariant (22). But in addition to these invariants the Painlevé cases (11)-(13) also have other invariants: Painlevé case (11) also has the invariants 4 and 6 of table 1, Painlevé case (12) also has invariant 6 of table 1 (5 drops out because it reduces to zero) and Painlevé case (13) also has invariant 5 of table 1 which in this case coincides with invariant 6.

Next, we point out that the idea that the existence of constants of the motion expresses some regularity of the motion is confirmed by the fact that most of the constants of the motion found only exist in a region of parameter space where $\sigma < b+1$, i.e. where the asymptotic motion is regular (fixed point or limit cycle). The exceptions are invariant 6 of table 1 and invariant (ii), which can also exist in the region $\sigma > b+1$.

On the limit cycle (7) xx^*, yy^* and z are constant. From this it follows that for F of the form (15) or (19) to be a constant its polynomial part must vanish on the limit cycle. This necessary condition was used as a test condition on our results.

Finally, we can conclude that as far as the Painlevé property and polynomial constants of the motion are concerned the complex Lorenz model is very similar to the real Lorenz model.

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